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The three-colouring problem as a special eight-vertex model

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Abstract. The equations giving the free energy per site and those governing the Bethe ansatz wavefunctions for a 'solid-on-solid' (SOS) model equivalent to the eight-vertex model are shown to be identical to those arising in the three-colouring problem for the particular choice of parameters $\eta = -\frac{1}{3}\mathrm{i}K'$, $v = 0$, using the original Baxter notation.

In the last twenty years considerable progress in two-dimensional statistical mechanics has been achieved through the exact solutions of many models, notably the six-vertex model (Lieb 1967) and various variants: the eight-vertex model (Baxter 1972a, 1973), the triangular three-spin model (Baxter and Wu 1974), the generalised hard-hexagon model (Baxter 1980) and recently its extension as a general SOS model (Andrews *et al* 1984). It is remarkable that the Ising model (Onsager 1944) and various free-fermion models (Fan and Wu 1968) as well as the previously mentioned models turn out to be special cases of the versatile eight-vertex model (Baxter and Enting 1976, Baxter 1982, Andrews *et al* 1984).

However, the three-colouring of the square lattice problem, although solved long ago (Baxter 1970), has not been connected to the eight-vertex model. Perhaps such a connection may be already known to Baxter who has solved both models, but as far as we know it has not appeared in the literature. It is the purpose of this paper to provide the technical details leading to the identification of the two models.

We proceed basically by choosing special values for the original parameters used by Baxter and show that the equations governing the Bethe ansatz eigenfunctions of the transfer matrix (Baxter 1973) are precisely those of the colouring problem of the square lattice with three colours as expounded in § 8.13 of Baxter (1982).

Let us first recall the definition of the eight-vertex model. We consider a square lattice with toroidal boundary conditions with an arrow covering of the bonds such that round a given site there can be only one of the eight possible configurations listed in figure 1 with their respective statistical weights:

$$a : b : c : d = \mathrm{sn}(v+\eta) : \mathrm{sn}(v-\eta) : \mathrm{sn}(2\eta) : k\,\mathrm{sn}(2\eta)\,\mathrm{sn}(v+\eta)\,\mathrm{sn}(v-\eta) \tag{1}$$

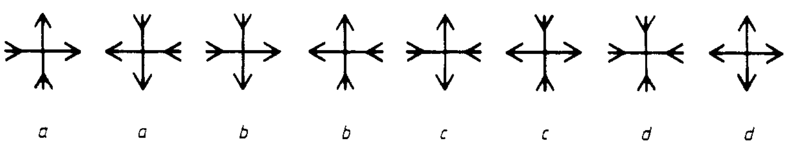

Figure 1. The eight arrow configurations allowed at a vertex.

where $\mathrm{sn}(x)$ is the Jacobian elliptic sine function of modulus k, $0<k<1$. The free energy per site of an infinite lattice can be calculated exactly, for example by solving an integral equation often arising in integrable systems (see e.g. equation (4.13) of Johnson *et al* 1973) as

$$f_{8\mathrm{v}}(\alpha, \lambda, k)=\tfrac{1}{2}(\lambda-\alpha-\mathrm{i}\pi)-\sum_{m=1}^{\infty}\frac{\sinh(\tau-\lambda)m\,\sinh(\lambda-\alpha)m}{m\sinh m\tau\cosh m\lambda} \tag{2}$$

where, K_k and K'_k being the complete elliptic integrals of moduli k and $k'=(1-k^2)^{1/2}$,

$$\alpha=-\mathrm{i}\pi(v/K_k) \qquad \lambda=-\mathrm{i}\pi(\eta/K_k) \qquad \text{and} \qquad \tau=\pi(K'_k/2K_k). \tag{3}$$

Now making the choices

$$\eta=\tfrac{1}{3}\mathrm{i}K'_k, \qquad v=0 \tag{4}$$

the free energy $f_{8\mathrm{v}}$ now takes the form:

$$f_{8\mathrm{v}}(0, \tfrac{2}{3}\tau, k)=(\tfrac{1}{3}\tau-\tfrac{1}{2}\mathrm{i}\pi)-\sum_{m=1}^{\infty}\frac{(r^{2m}-1)r^m}{m(1-r^m-r^{2m})(1-r^{2m})} \tag{5}$$

with $r=\exp(-\frac{2}{3}\tau)$. However, this is precisely the free energy of the three-colouring problem as given by equation (8.13.83) of Baxter (1982, hereafter called I) provided one relates the parameter τ to the product of the three activities of the colours by

$$\mathrm{i}\exp(\tfrac{2}{3}\tau)=z_1z_2z_3. \tag{6}$$

Moreover since the critical regime of the eight-vertex model is described by the parameter $\mu=\pi\lambda/\tau$ (Baxter 1972a), we have here $\mu=\frac{2}{3}\pi$ leading to a critical index $\alpha=\frac{1}{3}$, identical to that of the three-colouring problem†.

Before we go on with the proof, we should set the notation straight by relabelling the parameters η and k of § 8.13 of I by $\eta_{3\mathrm{c}}$, $k_{3\mathrm{c}}$ to distinguish them from those of the eight-vertex model. Also whenever Jacobian theta functions appear they shall be written as in Baxter (1972, hereafter called II), i.e. H_{Jb}, Θ_{Jb}, etc....

In II, it has been shown that the eight-vertex problem is equivalent to a covering of the plaquettes of the original square lattice by 'heights' l such that round a site there can be only one of the six height configurations depicted by figure 2. The corresponding statistical weights, with a global normalisation factor ρ', are expressed

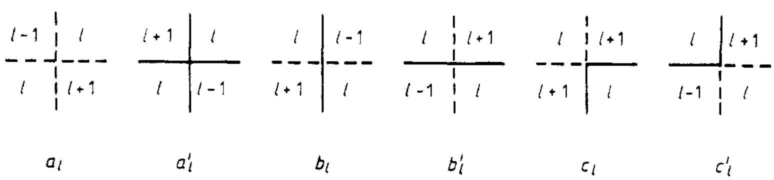

Figure 2. The six height configurations allowed at a vertex.

† This fact presumes already that the three-colouring problem is in fact a special eight-vertex model, and this is what will now be shown explicitly.

by

$$a_l = \rho' h(v+\eta) = a \qquad b_l = \rho' h(v-\eta)\frac{h(w_{l-1})}{h(w_l)} \qquad c_l = \rho' h(2\eta)\frac{h(w_l+\eta-v)}{h(w_l)h(w_{l+1})}$$

$$a'_l = \rho' h(v+\eta)\frac{h(w_{l+1})}{h(w_l)} \qquad b'_l = \rho' h(v-\eta) = b' \qquad c'_l = \rho' h(2\eta)h(w_l-\eta+v). \qquad (7)$$

This new notation is explained as follows:

$$w_l = w + 2l\eta \qquad \text{and} \qquad w = \text{arbitrary constant.}$$

The local height l varies from 1 to L, which is determined by

$$L\eta = 2m_1 K_k + \mathrm{i}m_2 K'_k. \qquad (8)$$

The elliptic function $h(u)$ is given in terms of Jacobian theta functions by

$$h(u) = H_{\mathrm{Jb}}(u)\Theta_{\mathrm{Jb}}(u)\exp\left(\mathrm{i}\frac{m_2\pi}{2K_k L\eta}(u^2+K_k^2)\right). \qquad (9)$$

Thus $h(u)$ is an entire function, odd and periodic of period $2L\eta$, with the property, for $m_1=0$ and $m_2=1$:

$$h(u+L\eta) = -h(u) \qquad (10)$$

(easily checked using the pseudo-periodicity of H_{Jb} and Θ_{Jb} under a shift $\mathrm{i}K'_k$ of the argument).

Since the arrangements of heights of figure 2 again obey the ice rule, it is natural to look for eigenfunctions of the transfer matrix: the Bethe ansatz wavefunctions. To keep the discussion as simple as possible we concentrate on the 'two-down' arrows (or particles) as given by (2.5) of II. Besides, if complete understanding of free theories are given by the one-particle structure, complete understanding of soluble systems by the Bethe ansatz is provided by the two-particle structure, as is well known.

The 'evolution' of two particles is described by figures 3(a) and (b). If we denote the two-particle wavefunction by $\tilde{f}(l|x_1, x_2)$ then it obeys the 'integral' equation,

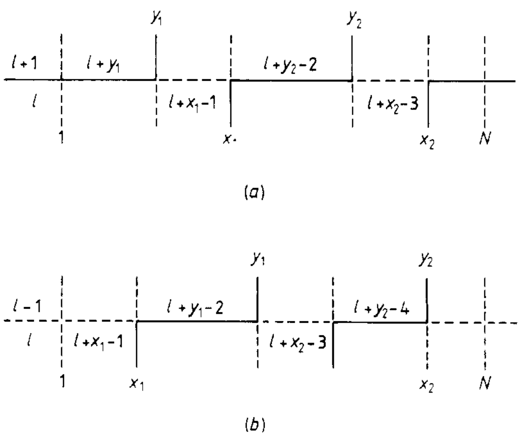

Figure 3. Graphical representations of the matrix elements $\tilde{D}_{\mathrm{L}}(l|X, Y)$ and $\tilde{D}_{\mathrm{R}}(l|X, Y)$ for two 'particles' or vertical lines separating consecutive decreasing heights on a row. Only heights to the left of the 'particles' are represented.

corresponding to N sites on a row

$$\Lambda_{8v}\tilde{f}(l|x_1,x_2)=r_{l+x_1}r_{l+x_2-2}\left(b'^N\sum_{y_1=1}^{x_1}\sum_{y_2=1}^{x_2}{}^{*}\,\tilde{D}_{\rm L}(l|X,Y)\tilde{f}(l+1|y_1,y_2)\right.$$
$$\left.+a^N\sum_{y_1=x_1}^{x_2}\sum_{y_2=x_2}^{N}{}^{*}\,\tilde{D}_{\rm R}(l|X,Y)\tilde{f}(l-1|y_1,y_2)\right). \tag{11}$$

The notation used in (11) is the following.

(i) The summation Σ^* excludes the cases $y_1=x_1=y_2$ and $y_1=x_2=y_2$.

(ii) Λ_{8v} is the two-particle eigenvalue of the transfer matrix.

(iii) $\tilde{D}_{\rm L}$ and $\tilde{D}_{\rm R}$ are matrix elements of the transfer matrix of an inhomogeneous six-vertex system with 'reduced' weights $\tilde{a}_l=\tilde{b}'_l=\tilde{c}_l=\tilde{c}'_l=1$ and $\tilde{a}'_l=p_l$, $\tilde{b}_l=q_l$,

$$\tilde{D}_{\rm L}(l|X,Y)=U(l+1|0,y_1,x_1)U(l-1|x_1,y_2,x_2)$$
$$\tilde{D}_{\rm R}(l|X,Y)=U(l-1|x_1,y_1,x_2)U(l-3|x_2,y_2,N+1)$$

where the function $U(l|x,y,x')$ is defined for $x<x'$, $x\leqslant y\leqslant x'$, as

$$U(l|x,y,x')=\begin{cases}q_{l+x-1} & \text{for } x=y\\ 1 & \text{for } x<y<x'\\ p_{l+x'-2} & \text{for } y=x'\end{cases}$$

and

$$p_l=\frac{h^2(v+\eta)h(w_{l-1})h(w_{l+1})}{h^2(2\eta)h(w_l-\eta-v)h(w_l+\eta+v)} \tag{12a}$$
$$q_l=\frac{h^2(v-\eta)h(w_{l-1})h(w_{l+1})}{h^2(2\eta)h(w_l-\eta+v)h(w_l+\eta-v)} \tag{12b}$$
$$r_l=\frac{h^2(2\eta)h(w_{l-1}-\eta+v)h(w_{l-1}+\eta-v)}{h(v+\eta)h(v-\eta)h(w_{l-2})h(w_{l-1})}. \tag{12c}$$

These are equations (2.8), (2.9), (4.11), (4.12) and (4.13) of II. Now taking $\tilde{f}(l|x_1,x_2)$ as a Bethe ansatz wavefunction, where a summation on all permutations P of the one-particle parameters is performed

$$\tilde{f}(l|x_1,x_2)=\sum_P A(P)\tilde{g}_{P_1}(l,x_1)\tilde{g}_{P_2}(l-2,x_2) \tag{13}$$

with the one-particle wavefunction, depending on a parameter u_j, $j=1,2$:

$$\tilde{g}_j(l-2j+2,x_j)=\exp(\mathrm{i}k'_jx_j)\frac{h(2\eta)}{h(v+\eta)}\frac{h(w_{l+x_j-1}+v-\eta)}{h(w_{l+x_j-2})}\frac{h(w_{l+x_j-1}-\eta-v-u_j)}{h(w_{l+x_j-1})}, \tag{14}$$

whose momentum is given by

$$\exp(\mathrm{i}k'_j)=\frac{h(v+\eta)}{h(v-\eta)}\frac{h(v+\eta+u_j)}{h(v-\eta+u_j)}. \tag{15}$$

One then finds a superposition amplitude

$$A(P)=\varepsilon(P)\prod_{j<m}h(u_{Pj}-u_{Pm}+2\eta) \tag{16}$$

and the eigenvalue

$$\Lambda_{8v}=[\rho'h(v-\eta)]^N\prod_{j=1}^{2}\frac{h(u_j-2\eta)}{h(u_j)}+[\rho'h(v+\eta)]^N\prod_{j=1}^{2}\frac{h(u_j+2n)}{h(u_j)} \tag{17}$$

(see equations (3.1), (4.29), (4.22), (4.26), (5.5) and (4.34) in II). The generalisation to n particles is a standard procedure.

We now particularise our choice of parameters given by (4) which implies that $L=3$ (because $m_1=0$ and $m_2=1$). Now using (10) we obtain a remarkable simplification for our equations since now $p_l=q_l=1$ (or $\tilde{a}_l=\tilde{a}'_l=\tilde{b}_l=\tilde{b}'_l=\tilde{c}_l=\tilde{c}'_l=1$ defines an ice model) and consequently $\tilde{D}_{\mathrm{L}}=\tilde{D}_{\mathrm{R}}=1$. Thus (11), (with N even) for the two-particle amplitude, has effectively the form of the two-particle amplitude obtained in the three-colouring problem (see equation (8.13.II) of I) when we make the following identifications.

(i) The colour σ is related to the height l (modulo 3) by

$$l=\sigma+1. \tag{18}$$

(ii) The one-particle wavefunction of the jth particle becomes

$$\tilde{g}_j(l-2j+2,x_j)=\exp(\mathrm{i}k_jx_j)\frac{h(w_{l-2j+2+x_j})}{h(w_{l-2j+2+x_j}+2\eta)}\frac{h(w_{l-2j+2+x_j}-u_j)}{h(w_{l-2j+2+x_j}-\eta)} \tag{19}$$

with $\exp(\mathrm{i}k_j)=h(\eta+u_j)/h(\eta-u_j)$. Since $l-2j+2=\sigma+j$ (mod 3) the eigenvalues $\Lambda_{8\mathrm{v}}$ and $\Lambda_{3\mathrm{c}}$ as well as the weights r_l and $\zeta(\sigma)=z_\sigma z_{\sigma+1}/(z_1z_2z_3)^{N/3}$ of the eight-vertex model and the three-colouring problem are connected by

$$\Lambda_{8\mathrm{v}}=(z_1z_2z_3)^{-N/3}\Lambda_{3\mathrm{c}} \qquad \text{and} \qquad r_{l-2j+2+x_j}=\zeta(x_j+j+\sigma). \tag{20}$$

We can now relate it to the one-particle wavefunction of the three-colouring problem (equation (8.13.13) of I): $\phi(x_j+j+\sigma)$ by

$$\tilde{g}_j(l-2j+2,x_j)=\exp[-\mathrm{i}k_l(\sigma+j)]\phi(x_j+j+\sigma) \tag{21}$$

where now

$$\phi(x_j+j+\sigma)=\exp[\mathrm{i}k_j(x_j+j+\sigma)]\frac{h(w_{x_j+j+\sigma})}{h(w_{x_j+j+\sigma}+2\eta)}\frac{h(w_{x_j+j+\sigma}-u_j)}{h(w_{x_j+j+\sigma}-\eta)} \tag{22}$$

has precisely the Bloch wavefunction form and the pseudo-periodicity

$$\phi(x_j+3)=\exp(\mathrm{i}3k_j)\phi(x_j) \tag{23}$$

due to the property of (10).

Putting the expression (21) back into (13), appropriately generalised to n particles (as in (31) of II)) we see that we obtain the coupling coefficient of the n particle in the three-colouring problem (see equation (8.13.13) of I)

$$A'(P)=A(P)\exp(-\mathrm{i}k-2\mathrm{i}k-\ldots-n\mathrm{i}k) \tag{24}$$

provided $\exp(-\mathrm{i}\sigma\,\Sigma_{j=1}^{n}k_j)=1$, which is the case in the ground state of the eight-vertex model (Johnson *et al* 1973).

To complete the identification of the two models we should also look at the two-particle phase shift due to scattering. Starting from the eight-vertex expression, now rewritten in terms of Jacobian theta functions

$$\frac{h(2\eta+u_1-u_2)}{h(2\eta-u_1+u_2)}=\exp\left(\frac{4\mathrm{i}\pi}{3K_k}(u_1-u_2)\right)\frac{h_{\mathrm{Jb}}(2\eta+u_1-u_2)}{h_{\mathrm{Jb}}(2\eta-u_1+u_2)} \tag{25}$$

where $h_{\mathrm{Jb}}(u)=H_{\mathrm{Jb}}(u)\Theta_{\mathrm{Jb}}(u)$.

Table 1. Special values of η characterising various soluble models.

Model	η	$\mu = \pi(\lambda/\tau)$	m_1	m_2	$L\eta = 2m_1K + im_2K'$
Hard hexagons	$\frac{1}{5}K$		1	0	10
General SOS	K/r		1	0	$2r$
Ising	$\frac{1}{4}iK'$	$\frac{1}{2}\pi$	0	1	4
Three-colouring	$\frac{1}{3}iK'$	$\frac{2}{3}\pi$	0	1	3
Triangular three-spin	$\frac{3}{8}iK'$	$\frac{3}{4}\pi$	0	3	8

Now recall that the elliptic integrals and moduli used in the three-colouring problem are $I(k_{3c})$ and $I'(k_{3c})$; however, they are connected to those of the eight-vertex by a Landen transformation because (see equation (8.13.77) of I and (3)), namely

$$\tau = \pi(K'_k/2K_k) = \pi(I'(k_{3c})/I(k_{3c})). \tag{26}$$

Thus, according to appendix B of Johnson *et al* (1973) we can identify

$$h_{Jb}(u, k) = \text{constant} \times H_{Jb}(\boldsymbol{\alpha}, k_{3c}) \tag{27}$$

with a rescaling of the argument

$$u = (2K_k/I(k_{3c}))\boldsymbol{\alpha}$$

and the coupling constant

$$2\eta = \eta_{3c}(2K_k/I(k_{3c})). \tag{28}$$

This ultimately leads to the phase shift of equations (8.13.36) and (8.13.73a, b) obtained in the three-colouring problem by Baxter.

We close by giving a table illustrating the connections of various soluble models to the eight-vertex models showing the outstanding role of the integer L, with respect to the parameter η of Baxter. We note that (Baxter and Wu 1973) the triangular three-spin model can be treated also as a colouring problem of the triangular lattice. A final remark concerns the associated XYZ chain (Baxter 1972b)

$$\mathscr{H}_{XYZ} = \sum_i (J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y + J_z\sigma_i^z\sigma_{i+1}^z) - \text{constant} \tag{29}$$

which has a threefold symmetry because there is a relation

$$J_xJ_y + J_yJ_z + J_zJ_x = 0 \tag{30}$$

defining a cone passing through the origin. Baxter has studied the part $(J_x + J_y + J_z) > 0$ associated with $\mu = \frac{1}{3}\pi$. We now see that the part $(J_x + J_y + J_z) < 0$ associated with $\mu = \frac{2}{3}\pi$ is related to the three-colouring problem. Perhaps this threefold symmetry might be the origin of a self-triality of the XYZ chain as conjectured before (Shankar 1981).

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